

# Theoretical model of longitudinal relaxation oscillations induced by HOMs\*

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## Abstract

Taking advantage of the vastly different time scales of the problem, a simple analytical model of the relaxation oscillations has been developed. First a continuous approximation of the impulsive discrete forces is made. Then only the synchronous components of the force are retained to describe the slowly varying amplitude and frequency of the relaxation oscillation. A two particle version of this model reproduces the main characteristics of the system. A more complete paper is in preparation [1].

## 1 ANALYTICAL MODEL

### 1.1 Continuous Approximation

A particle in a storage ring generates a electromagnetic field, a wakefield, that acts back on itself. The wakefield of a cavity can be represented as the impulse response of a narrow-band resonator of resistance,  $R_S$ , frequency,  $\omega_R$ , and damping factor  $\alpha_R$ .

$$W(t - \tau) \cong U(\tau) 2\alpha_R R_S e^{-\alpha_R(t-\tau)} \cos \omega_R(t - \tau)$$

The wake seen by any particle is the sum of all wakes generated by it and all other particles in all previous turns. For a single bunch, the decelerating wake potential seen at time,  $t$ , is then

$$W(t) = 2\alpha_R R_S \sum_{u=-\infty}^t e^{-\alpha_R(t-u)} \cos \omega_R(t - u),$$

The arrival times of the source particle generating the wakefield,  $u$ , and the particle experiencing the wakefield,  $t$ , are expressed by  $t = nT_0 + \tau_n$  and  $u = kT_0 + \tau_k$ , where  $\tau_n, \tau_k \ll T_0$ . Representing  $\omega_R = p\omega_0 + \omega_z$  as an integral multiple and a fractional part of the revolution harmonic, the sum becomes

$$W(t) \cong 2\alpha_R R_S \frac{1}{T_0} \int_{(k-1)T_0}^{kT_0} \sum_{n=-\infty}^n e^{-\alpha_R(n-k)T_0} \times \cos(\omega_z(n-k)T_0 + \omega_R(\tau_n - \tau_k)) du$$

When the bunch has  $N$  particles of charge,  $e$ , giving a machine current,  $I$ , the electrical potential  $V(t)$  generated by the wake is

$$V(t) = 2\alpha_R R_S I \int_{-\infty}^t e^{-\alpha_R(t-u)} \times \cos(\omega_z(t-u) + \omega_R(\tau(t) - \tau(u))) du$$

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### 1.2 Evaluation of the Integral

The continuous approximation of the synchrotron motion is that of common pendulum motion, for which oscillations as large as  $\pi/2$  are still very sinusoidal. Therefore  $\tau(t)$  and  $\tau(u)$  can be represented as

$$\tau_t = \hat{\tau}_t \cos(\omega_{st}t + \phi_t); \quad \tau_u = \hat{\tau}_u \cos(\omega_{su}t + \phi_u)$$

with  $\hat{\tau}_t, \hat{\tau}_u, \omega_{st}, \omega_{su}, \phi_t$ , and  $\phi_u$  all slowly varying compared to the synchrotron period. The exponential damping in the integral means that only important contributions come from times no further back than a few damping times, during which these quantities can be treated as constant. With the notation  $r_t = \hat{\tau}_t \omega_R$ ,  $r_u = \hat{\tau}_u \omega_R$ , the integral in the driving term can be expressed as the real part of

$$\sum_{p,k=-\infty}^{\infty} \frac{j^{p-k} J_p(r_t) J_k(r_u)}{\alpha_R + j(k\omega_{su} - \omega_z)} e^{j(k(\omega_{su} + \phi_u) + p(\omega_{st} + \phi_t))t}$$

### 1.3 Application of KBM Method

The averaging method of Krylov, Bogoliubov, and Mitropolskii [2] [3] is well suited to such an oscillatory problem with slowly varying parameters [4]. To solve a driven harmonic oscillator,  $\ddot{x} + \omega_{so}^2 x = f_x(x, \dot{x})$ , new variables,  $(r(t), \phi(t))$ , are defined in terms of  $(x(t), \dot{x}(t))$  by

$$x = r \cos(\omega_{so}t + \phi); \quad \dot{x} = -\omega_{so} r \sin(\omega_{so}t + \phi)$$

The averaged evolution equations of the oscillation amplitude and phase become

$$\dot{r} = -\frac{1}{2\omega_{so}} F_{S1}(\bar{r}, \bar{\phi}); \quad \dot{\phi} = -\frac{1}{2\omega_{so}\bar{r}} F_{C1}(\bar{r}, \bar{\phi})$$

$F_{S1}$  and  $F_{C1}$  are the Fourier coefficients of the frequency  $\omega_{su}$  in the wakefield generated by the particle at  $(r_u, \phi_u)$  on the particle at  $(r_t, \phi_t)$

$$F_{S1} = -2A\alpha_R R_S I \sum_{k=1}^{\infty} J_k(r_u) [J_{k-1}(r_t) + J_{k+1}(r_t)] \times \quad (1)$$

$$[(a_k^- - a_k^+) \cos(k\Delta\phi) - (b_k^- - b_k^+) \sin(k\Delta\phi)]$$

$$F_{C1} = 2A\alpha_R R_S I \{2b_0^+ J_0(r_u) J_1(r_t) \quad (2)$$

$$+ \sum_{k=1}^{\infty} J_k(r_u) [J_{k-1}(r_t) - J_{k+1}(r_t)] \times$$

$$[(b_k^- - b_k^+) \cos(k\Delta\phi) + (a_k^- - a_k^+) \sin(k\Delta\phi)]\}$$

where  $A = \frac{\omega_{s0}^2}{V_{RF} |\cos \phi_s|}$ ,  $\Delta\phi = \phi_t - \phi_u$  and

$$a_k^\pm = \frac{\alpha_R}{\alpha_R^2 + (k\omega_{su} \pm \omega_z)^2}; \quad b_k^\pm = \frac{(k\omega_{su} \pm \omega_z)}{\alpha_R^2 + (k\omega_{su} \pm \omega_z)^2}$$

This paper discusses only the case when  $\omega_R = \omega_{RF}$ . Extensions to other values of  $\omega_R$  are straightforward.

The wakefield is not the only perturbation to the harmonic equation. A radiation damping term,  $-\alpha_{rad} \bar{r}_t$ , contributes to the  $\dot{\bar{r}}_t$  equation. The amplitude dependent decrease in pendulum frequency can be approximated by a term quadratic in  $\bar{r}_t$  [3]. The KBM method is applied by treating the three terms as independent contributions to the equations of motion of  $\bar{r}$  and  $\bar{\phi}$ , giving the final, averaged equations of motion for a test particle at  $(r_t, \phi_t)$  due to a macroparticle at  $(r_u, \phi_u)$

$$\dot{\bar{r}} = -\frac{1}{2\omega_{st}} F_{S1}(\bar{r}_t, \bar{\phi}_t, \bar{r}_u, \bar{\phi}_u) - \alpha_{rad} \bar{r}_t \quad (3)$$

$$\dot{\bar{\phi}} = -\frac{1}{2\omega_{st} \bar{r}_t} F_{C1}(\bar{r}_t, \bar{\phi}_t, \bar{r}_u, \bar{\phi}_u) - \frac{1}{16} \bar{r}_t^2 \omega_{so} \quad (4)$$

## 2 ANALYSIS OF RELAXATION OSCILLATIONS:

### 2.1 Properties of Wakefield Terms

The convenient reference frame for these equations is one rotating in phase with  $\tau_u$ , and in which  $\tau_u$  moves radially along the  $\phi = 0$  axis. The angular position of  $\tau_t$  is the difference in phase,  $\Delta\phi$ , between it and the source. As  $\tau_t$  rotates in this frame, the forces from  $\tau_u$  change character, from damping to anti-damping, and from frequency increase to frequency decrease. For narrow band resonators tuned with  $\omega_z = \omega_{su}$ , the line of maximal growth and zero frequency shift both lie close to  $\Delta\phi = 0$ .

### 2.2 Linear regime

For the case of a single macroparticle model,  $r_t = r_u$ , and  $\Delta\phi = 0$ . In most cases, the  $m = 0$  and  $m = 1$  terms of the series give a good approximation to the total force. Using the narrowband resonator impedance approximation, for small  $r$ , after defining  $Z_k^p = Z(p\omega_0 + k\omega_z)$ , one recovers the formulae for growth and frequency shifts given in references[5][6].

$$\dot{\bar{r}} = \frac{\omega_{s0}}{2V_{RF} |\cos \phi_s|} I \operatorname{Re} \{ Z_1^h - Z_{-1}^h \} \bar{r} - \alpha_{rad} \bar{r}$$

$$\dot{\bar{\phi}} = \frac{\omega_{s0}}{2V_{RF} |\cos \phi_s|} I \operatorname{Im} \{ 2Z_0^h - Z_1^h - Z_{-1}^h \}$$

### 2.3 Growth as a Macroparticle

Equations 1 and 2 give the driving force acting on a test particle  $(r_t, \Delta\phi)$  produced by the main body  $(r_u, \Delta\phi = 0)$ . In expansions around the main body,  $\Delta r$  is defined as  $\Delta r \triangleq r_t - r_u$ .

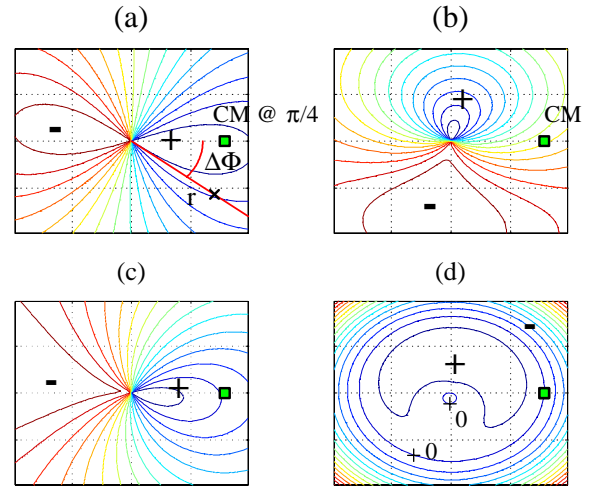


Figure 1:  $\dot{\bar{r}}, \dot{\bar{\phi}}$  terms in  $(r_t, \phi_t)$  plane. (a)  $F_{S1}$ ; (b)  $F_{C1}$ ; (c)  $F_{S1}$  and  $\alpha_{rad} \bar{r}$ ; (d)  $F_{C1}$  and pendulum shift

Figure 1 shows the effect of the growth and frequency shift due to the macroparticle wake in the rotating frame:

- particles ahead of (behind) the main body ( $r_u = 0, \Delta\phi > 0$  ( $\Delta\phi < 0$ )) experience a lesser (greater)  $\dot{\bar{\phi}}$  than the main body and will fall back (catch up) to it.(figure 1(b))
- particles at  $(\Delta r < 0$  ( $\Delta r > 0$ ),  $\Delta\phi = 0$ ) feel more (less) growth than the main particle and will grow (fall) toward it (figure 1(a),(c))

This justifies why the bunch keeps its cohesion during its growth [7]; the main body is an attractor for all the particles of the bunch. As the oscillation increases to moderate amplitudes, two nonlinear effects become important: the Bessel terms decrease the growth rate; the pendulum frequency shift starts to dominate the frequency term.

### 2.4 Filamentation

The pendulum frequency shift now causes a strong enough asymmetry in  $\Delta r$  that the test particles start to escape from the front of the bunch.

- Particles with  $\Delta r = 0$  experience the same growth as the main body, and will tend to group back towards it as during the growth.
- Particles with  $(\Delta r > 0, \Delta\phi = 0)$  slow down more than the main body and acquire a  $\Delta\phi < 0$ . This leads them to an area of smaller radial growth, decreasing  $\Delta r$ , and hence increasing frequency. This sequence leads back to the main body.
- But particles with  $(\Delta r < 0, \Delta\phi = 0)$  speed up more than the main body and acquire a  $\Delta\phi > 0$ . Once they cross the angle of maximal growth, the particles

at positive  $\Delta\phi$  experience a smaller driving force and move to even more negative  $\Delta r$ . The particles escape from the front of the bunch.

The experimental data shows the decrease in strength of the growth term, only part of which comes from the weaker Bessel terms. The bunch saturates at lower amplitudes than this effect predicts. Streak camera images reveal the loss of particles [7]. The relaxation of the oscillation comes from the loss of growth due to the leakage of particles away from the main body and the formation of a second attractor close to the center of phase space.

## 2.5 Damping of system

As the escaping particles spiral away from the main body towards the center, they alternately experience positive and negative forces from the main body. Over a rotation of  $\Delta\phi = 2\pi$ , the net growth due to the main body nearly vanishes, so the particles damp at about the radiation damping rate. The only growth they see is due to other particles synchronous to them.

The finite main body amplitude,  $\tau_u \neq 0$ , implies equations 1 and 2 are non-zero at the origin; therefore all values of  $\dot{\phi}$  exist near there (equation 4). On the locus of points in phase with the main body, the particles will again feel the main body's wake. On the locus exists a point at which the radial growth vanishes; very close by is an attractor for particles leaving the main body. To experience no growth from the wake,  $\Delta\phi \sim \pi/2$  when  $\omega_z = \omega_{so}$  (figure 1). As charge accumulates at this point, its self-generated wake increases in strength and  $\Delta\phi$  of the stable point increases to acquire damping from the main body. The second attractor is actually not fixed, but grows slowly in amplitude, attracting more and more particles until it becomes the new main body in the next relaxation cycle.

## 2.6 Visualization of second attractor

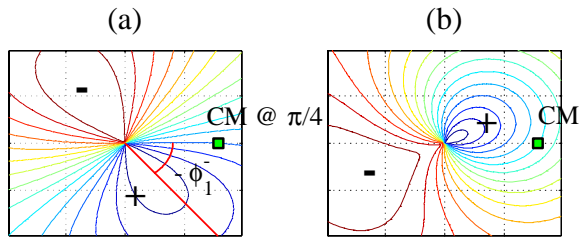


Figure 2:  $\dot{r}, \dot{\phi}$  terms for  $\omega_z > \omega_{su}$ . (a)  $F_{S1}$ ; (b)  $F_{C1}$ ;

An interesting case is observed when the lower edge of the resonance coincides with the synchrotron sideband ( $\omega_z > \omega_{so}$ ) [7]. The second attractor forms away from the center nearly  $\pi$  out of phase with the main initial body and the system damps much more slowly than when  $\omega_z = \omega_{so}$ . Defining  $\phi_k^\pm \triangleq \arctan[(k\omega_{su} \pm \omega_z)/\alpha_R]$ , the line of maximal growth is now approximately  $|\phi_1^-|$  ahead of the

main body (figure 2). Test particles now trying to escape from the front have a more difficult time than for the case  $\omega_z = \omega_{so}$  for two reasons:

- as  $\Delta\phi$  of the particles increase, the particles move closer to the line of maximal growth, grow radially and slow down;
- they need to precess  $\Delta\phi = 2|\phi_1^-|$  before their radial growth is less than that of the macroparticle.

Longer escape times mean longer damping times of the relaxation oscillation for  $\omega_z > \omega_{so}$ .

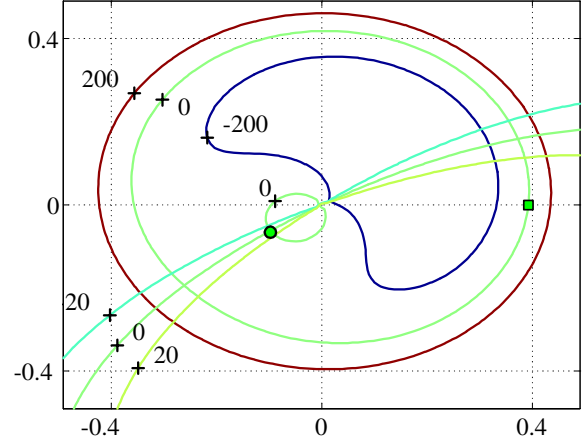


Figure 3: Attractor location for  $\omega_z > \omega_{su}$ . Initial main body at  $(0, .4)$ . Second attractor around  $(-.1, -.06)$

In this case the second attractor starts at  $\phi \sim \pi/2 + |\phi_1^-|$  and increases in phase towards  $\pi$  as it gains particles and grows in amplitude (figure 3).

## 3 REFERENCES

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